# Unique maximizers may not be well-separated

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Let $\Theta$ be a topological space and let $Q : \Theta \to \mathbb{R}$. $Q$ has a **well-separated point of maximum** at $\theta_0 \in \Theta$ if

$$\text{for any open } G \subseteq \Theta \text{ with } \theta_0 \in G, \quad Q(\theta_0) > \sup_{\theta \in \Theta \setminus G} Q(\theta). \tag{1}$$

The following example fails (1): with $\Theta = [-1, 1]$,

$$Q_1(\theta) = \begin{cases} 0 & \text{if } |\theta| = 1/(2k\pi) \text{ for some } k \in \mathbb{N}, \\ 1 & \text{if } \theta = 0, \\ \cos(1/\theta) & \text{otherwise.} \end{cases}$$

$Q_1$ has a unique maximizer at 0, but given any $0 < \varepsilon \leq 1/2\pi$, $\sup_{\varepsilon \leq |\theta| \leq 1} Q_1(\theta) = 1 = Q_1(0)$.

$Q_1$ exploits oscillatory behavior of the cosine together with discontinuities to achieve the desired failure. If we impose continuity, we would need non-compactness of $\Theta$ to induce a failure. To that end, let $\Theta = \mathbb{R}$ and

$$Q_2(\theta) = (1 - |\theta|)\mathbf{1}\{|\theta| \leq 1\} + \left(\frac{2}{1 + e^{1-|\theta|}} - 1\right) \cdot \mathbf{1}\{|\theta| > 1\}.$$

For any $\varepsilon \in (0, \infty)$, $\sup_{|\theta| \geq \varepsilon} Q_2(\theta) = 1 = Q_2(0)$, even though $Q_2$ is continuous and has a unique maximizer at 0.

In $M$-estimation, $Q$ typically denotes the population objective function. Condition (1) is often assumed to establish consistency of an $M$-estimator; see, for example, Corollary 3.2.3(i) of van der Vaart and Wellner (2023, p. 395). Other results provide primitive sufficient conditions for (1). For instance, Theorem 2.1 of Newey and McFadden (1994, p. 2121) assumes compact $\Theta$ and continuous $Q$, which together imply condition (1).

# References


Newey, Whitney K., and Daniel McFadden. 1994. "Chapter 36: Large sample estimation and hypothesis testing," 2111–2245. Handbook of Econometrics, Volume 4. Elsevier.

van der Vaart, Aad, and Jon Wellner. 2023. *Weak Convergence and Empirical Processes: With Applications to Statistics.* 2nd ed. Springer Series in Statistics. Springer New York.