A note on asymptotic normality of sample quantiles

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1 Introduction

This note revisits a classic asymptotic normality result for sample quantiles which has been known since Mosteller (1946) and Smirnov (1949). The approach given here follows those of Pollard (1984, Chapter III Section 4 Example 24, p. 53) and Serfling (1980, Section 2.3.3, pp. 77-84). In particular, we relate the sampling distribution of the sample quantile to probabilities determined by binomial sums whose probability parameters may drift with the number of trials. This drift phenomenon motivates the use of the classic central limit theorem (CLT) for independent triangular arrays due to Liapunov.

Let $\{Y_i\}_{i=1}^{\infty}$ be a sequence of independent and identically distributed (iid) real valued random variables on a common probability space $(\Omega, \mathscr{F}, \Pr)$. Denote their common cumulative distribution function (cdf) [also called the population cdf] by

$$F(y) := \Pr\{Y_1 \leq y\}.$$

The parameter of interest are the associated population quantiles, which are

$$\theta_p := Q(p; F) := \inf\{y \in \mathbb{R} : F(y) \geq p\}, \quad \text{for } p \in (0, 1). \tag{1.1}$$

The empirical cdf (ecdf) corresponding to the first n observations is

$$\widehat{F}_n(y) := \widehat{F}_n\left(y; Y_1, \ldots, Y_n\right) := \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}\left\{Y_i \leq y\right\},$$

with corresponding sample quantiles

$$\widehat{\theta}_{p,n} := \inf\left\{y \in \mathbb{R} : \widehat{F}_n(y) \geq p\right\} = Q\left(p; \widehat{F}_n\right) \quad \text{for } Q \text{ defined in (1.1).} \tag{1.2}$$

We are interested in limits of the sampling distribution

$$H_{p,n}(t) := \Pr\left\{\sqrt{n}\left(\widehat{\theta}_{p,n} - \theta_p\right) \leq t\right\} = \Pr\left\{\widehat{\theta}_{p,n} \leq \theta_p + \frac{t}{\sqrt{n}}\right\}. \tag{1.3}$$

To characterize limits of $H_{p,n}(\cdot)$ in (1.3), we will sometimes need directional differentiability assumptions on F. These are defined in Definition 1.1 below. Theorem 1.1 provides a statement of the main asymptotic normality result. Corollary 1.1 states the more commonly known form of the result for absolutely continuous F with density f continuous at θ_p.

Definition 1.1. The function $F : \mathbb{R} \to \mathbb{R}$ is right-differentiable at y with right-derivative $f(y)$ iff for every real sequence $\{\delta_n\}$ such that $\delta_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \delta_n = 0$,

$$\lim_{n\to\infty} \frac{F\left(y + \delta_n\right) - F(y)}{\delta_n} = f(y). \tag{1.4}$$

The function F is left-differentiable at y with left-derivative $f(y)$ if (1.4) holds instead for every real sequence $\{\delta_n\}$ such that $\delta_n < 0$ for all $n \in \mathbb{N}$ and and $\lim_{n\to\infty} \delta_n = 0$.

Remark 1. F is differentiable at y with derivative $f(y)$ iff F is both right- and left-differentiable as in Definition 1.1 and its left- and right-derivatives are both equal to $f(y)$.

Theorem 1.1. *Suppose the population cdf F, and $p \in (0,1)$ and $\theta_p \in \mathbb{R}$ satisfy $p = F\left(\theta_p\right)$. Then*

$$\lim_{n\to\infty} H_{p,n}(0) = \frac{1}{2}. \tag{1.5}$$

Suppose in addition to the hypothesis $p = F(\theta_p)$, one of the following holds:

i. F is right-differentiable at θ_p with right-derivative $f(\theta_p)$, and $t > 0$;

ii. F is left-differentiable at θ_p with left-derivative $f(\theta_p)$, and $t < 0$.

Then

$$\lim_{n\to\infty} H_{p,n}(t) = \Phi\left(\frac{f(\theta_p)\, t}{\sqrt{p(1-p)}}\right). \tag{1.6}$$

Therefore if in addition to the hypothesis $p = F(\theta_p)$, F is differentiable at θ_p with derivative $f(\theta_p)$, then

$$\sqrt{n}\left(\widehat{\theta}_{p,n} - \theta_p\right) \rightsquigarrow \mathrm{N}\left(0, \sigma^2(p)\right) \quad \textit{as } n \to \infty, \ \textit{where } \sigma^2(p) := \frac{p(1-p)}{f(\theta_p)^2}. \tag{1.7}$$

Corollary 1.1. *If F is absolutely continuous with Lebesgue-density f and f is continuous at θ_p, then (1.7) also holds, i.e.*

$$\sqrt{n}\left(\widehat{\theta}_{p,n} - \theta_p\right) \rightsquigarrow \mathrm{N}\left(0, \sigma^2(p)\right) \quad \textit{as } n \to \infty, \ \textit{where } \sigma^2(p) := \frac{p(1-p)}{f(\theta_p)^2}.$$

The rest of this note is organized as follows. Corollary 1.1 is proven in Section 2. Theorem 1.1 is proven in Section 3. Section 4 states and proves some fundamental properties of cdfs that are used throughout these sections. Section 5 proves a key central limit theorem (Theorem 3.1) used in the proof of Theorem 1.1.

2 Proof of Corollary 1.1

Corollary 1.1 arises as a consequence of (1.7) in Theorem 1.1. To that end, we are done if we show that $p = F(\theta_p)$ and that F is differentiable at θ_p with derivative equal to $f(\theta_p)$. Absolute continuity of F implies its continuity on $\mathbb{R}$ and hence, at θ_p. Lemma 4.3 shows that continuity of F at its p-quantile θ_p implies $F(\theta_p) = p$. Hence, we are left to show differentiability of F at θ_p with derivative $f(\theta_p)$. This is implied by continuity of the Lebesgue density f at θ_p.[1] To see this note that by definition of f as a Lebesgue density, given $a, b \in \mathbb{R}$ such that $a \leq \theta_p \leq b$,

$$F(b) - F(a) = \int_a^b f(x)\, \mathrm{d}x.$$

Let $\delta = \delta(\varepsilon, \theta_p) > 0$ be such that $|f(y) - f(\theta_p)|$ if $|y - \theta_p| < \delta$. Let a, b satisfy $\theta_p - \delta < a \leq \theta_p \leq b < \theta_p + \delta$ and $a < b$. Then

$$\left| \frac{F(b) - F(a)}{b - a} - f(\theta_p) \right| = \left| \frac{\int_a^b f(x); \mathrm{d}x}{b - a} - f(\theta_p) \right| = \left| \frac{1}{b - a} \int_a^b \left(f(x) - f(\theta_p) \right)\, \mathrm{d}x \right| < \varepsilon.$$

□

3 Proof of Theorem 1.1

Differentiability of F at θ_p with derivative $f(\theta_p)$ implies both left- and right-differentiability with left- and right-derivatives both equal to $f(\theta_p)$ (see Remark 1). Hence (1.7) is implied by the combination of (1.5) and (1.6). So it suffices to prove (1.5) and (1.6). To start, recall from (1.3) that

$$H_{p,n}(t) = \Pr\left\{ \widehat{\theta}_{p,n} \leq \theta_p + \frac{t}{\sqrt{n}} \right\}.$$

We start with a lemma which allow us to relate this to the ecdf $\widehat{F}_n$.

Lemma 4.1 shows in (4.1) that $\widehat{\theta}_p \leq y$ if and only if $\widehat{F}_n(y) \geq p$. Combine this with (1.3) by setting $y = \theta_p + (t/\sqrt{n})$ to see that

$$H_{p,n}(t) = \Pr\left\{ \widehat{F}_n\left(\theta_p + \frac{t}{\sqrt{n}} \right) \geq p \right\}.$$

For brevity, rewrite this as

$$H_{p,n}(t) = \Pr\left\{ \widehat{F}_n\left(y_{p,n}(t) \right) \geq p \right\}, \quad \text{where} \quad y_{p,n}(t) = \theta_p + \frac{t}{\sqrt{n}}. \tag{3.1}$$

1. The proof offered here is essentially that given for one part of the Fundamental Theorem of Calculus in Rudin (1976, Theorem 6.20, pp. 133-134).

We know that $n\widehat{F}_n(y_{p,n}(t)) \sim \text{Binomial}(n, F(y_{p,n}(t)))$. To help characterize limits of $H_{p,n}$ via (3.1), Theorem 3.1 provides a central limit theorem for Binomial distributions whose trial success probabilities can potentially change with the number of trials.

Theorem 3.1. *Let $Z \sim \mathrm{N}(0,1)$, $\Phi(z) := \Pr\{Z \leq z\}$, $\{p_n\}$ be a sequence in $[0,1]$ and $S_n \sim \text{Binomial}(n, p_n)$.*

$$\textit{If} \quad \lim_{n\to\infty} \frac{1}{\sqrt{np_n(1-p_n)}} = 0, \quad \textit{then} \quad \frac{S_n - np_n}{\sqrt{np_n(1-p_n)}} =: Z_n \rightsquigarrow Z. \tag{3.2}$$

Hence, if for some $p_* \in (0,1)$ and $\Delta \in \mathbb{R}$, $p_n = p_* + n^{-1/2}\Delta + o\left(n^{-1/2}\right)$, then

$$\lim_{n\to\infty} \Pr\left\{\frac{S_n}{n} \geq p_*\right\} = \Phi\left(\frac{\Delta}{\sqrt{p_*(1-p_*)}}\right). \tag{3.3}$$

Proof. See Section 5. □

3.1 Proof of (1.5) in Theorem 1.1

In this case, $t = 0$ and so $y_{p,n}(t) = \theta_p$ is constant in n. Set

$$S_n = n\widehat{F}_n(\theta_p) \quad \text{and} \quad p_n = F(\theta_p). \tag{3.4}$$

The hypotheses for (3.2) and (3.3) are satisfied with $p_n = p_* = F(\theta_p)$ and $\Delta = 0$. Furthermore, since $p = F(\theta_p)$ by hypothesis to (1.5), from (3.1), (3.3), and (3.4) we get $\lim_{n\to\infty} H_{p,n}(0) = \Phi(0) = \frac{1}{2}$ which proves (1.5). □

3.2 On necessity of the condition $p = F(\theta_p)$

Continue to consider the case $t = 0$, but suppose that $p \neq F(\theta_p)$. By Lemma 4.2, $p \leq F(\theta_p)$. So if $p \neq F(\theta_p)$, then it must be the case that $p < F(\theta_p)$. We can still apply (3.2) in Theorem 3.1 still applies with $S_n = n\widehat{F}_n(\theta_p)$ and $p_n = F(\theta_p)$. Then combine (3.1) and (3.2) as follows:

$$
\begin{aligned}
H_{p,n}(0) = &\Pr\left\{\widehat{F}_n\left(\theta_p\right) \geq p\right\} = \Pr\left\{\frac{n\left(\widehat{F}_n\left(\theta_p\right) - F\left(\theta_p\right)\right)}{\sqrt{nF\left(\theta_p\right)\left(1 - F\left(\theta_p\right)\right)}} \geq \frac{n\left(p - F\left(\theta_p\right)\right)}{\sqrt{nF\left(\theta_p\right)\left(1 - F\left(\theta_p\right)\right)}}\right\} \\
= &\Pr\left\{Z_n \geq \frac{\sqrt{n}\left(p - F\left(\theta_p\right)\right)}{\sqrt{F\left(\theta_p\right)\left(1 - F\left(\theta_p\right)\right)}}\right\}.
\end{aligned}
$$

But by $p < F\left(\theta_p\right)$,

$$
\lim_{n\to\infty} \frac{\sqrt{n}\left(p - F\left(\theta_p\right)\right)}{\sqrt{F\left(\theta_p\right)\left(1 - F\left(\theta_p\right)\right)}} = -\infty,
$$

and so

$$
\lim_{n\to\infty} H_{p,n}(0) = \lim_{n\to\infty} \Pr\left\{Z_n \geq \frac{\sqrt{n}\left(p - F\left(\theta_p\right)\right)}{\sqrt{F\left(\theta_p\right)\left(1 - F\left(\theta_p\right)\right)}}\right\} = 1.
$$

Of course since $1 \geq H_{p,n}(t) \geq H_{p,n}(0)$ if $t > 0$, $\lim_{n\to\infty} H_{p,n}(t) = 1$ for $t > 0$ as well. That is, if $p \neq F\left(\theta_p\right)$, we cannot get a non-degenerate Gaussian limit for $\sqrt{n}\left(\widehat{\theta}_{p,n} - \theta_p\right)$.

Since we focus only on the cases where $H_{p,n}$ limits to a non-degenerate Gaussian, it is therefore necessary to impose the restriction $p = F\left(\theta_p\right)$. To ensure this holds for all $p \in (0, 1)$ we can maintain an assumption of continuity of F at θ_p as in Lemma 4.3.

3.3 Proof of (1.6) in Theorem 1.1

Here we set

$$
S_n = n\widehat{F}_n\left(y_{p,n}(t)\right), \quad p_n = F\left(y_{p,n}(t)\right), \quad p_* = p = F\left(\theta_p\right), \quad \text{and} \quad \Delta = f\left(\theta_p\right)t. \tag{3.5}
$$

Then (3.1) becomes

$$
H_{p,n}(t) = \Pr\left\{\frac{S_n}{n} \geq p_*\right\}.
$$

Hence we wish to apply (3.3) and so we have to show p_n, p_* and Δ as defined in (3.5) satisfy

$$\lim_{n\to\infty} \sqrt{n}\left(p_n - p_* - \frac{\Delta}{\sqrt{n}}\right) = 0. \tag{3.6}$$

To that end, let $\delta_n = t/\sqrt{n}$ and note that since $t \neq 0$,

$$\begin{aligned} \sqrt{n}\left(p_n - p_* - \frac{\Delta}{\sqrt{n}}\right) &= \sqrt{n}\left(F\left(\theta_p + \delta_n\right) - F\left(\theta_p\right) - f\left(\theta_p\right)\delta_n\right) \\ &= t \cdot \frac{F\left(\theta_p + \delta_n\right) - F\left(\theta_p\right) - f\left(\theta_p\right)\delta_n}{\delta_n}. \end{aligned}$$

Therefore, under either hypothesis of directional differentiability (since the sign of δ_n and t are equal),

$$\lim_{n\to\infty} \sqrt{n}\left(p_n - p_* - \frac{\Delta}{\sqrt{n}}\right) = t \lim_{n\to\infty} \cdot \frac{F\left(\theta_p + \delta_n\right) - F\left(\theta_p\right) - f\left(\theta_p\right)\delta_n}{\delta_n} = 0,$$

which shows (3.6). Apply (3.3) with (3.5) to get

$$\lim_{n\to\infty} H_{p,n}(t) = \lim_{n\to\infty} \Pr\left\{\frac{S_n}{n} \geq p_*\right\} = \Phi\left(\frac{\Delta}{\sqrt{p_*\left(1-p_*\right)}}\right) = \Phi\left(\frac{f\left(\theta_p\right)t}{\sqrt{p(1-p)}}\right),$$

which is exactly (1.6). □

4 Some basic results on the behavior cdfs

Lemma 4.1. *Given* $(p, y) \in (0,1) \times \mathbb{R}$*,*

$$p \leq F(y) \iff Q(p;F) \leq y. \tag{4.1}$$

Proof of Lemma 4.1. $p \leq F(y) \implies Q(p;F) \leq y$ is immediate from the definition of Q. By Lemma 4.2 $p \leq F(Q(p;F))$ and so if $Q(p;F) \leq y$ then $p \leq F(Q(p;F)) \leq F(y)$. □

Lemma 4.2. *For every* $p \in (0,1)$*,* $F(Q(p;F)) \geq p$*.*

Proof of Lemma 4.2. Suppose that for some $p \in (0,1)$, $F(Q(p;F)) < p$. Take any sequence $\{y_n\}$ such that $Q(p;F) < y_{n+1} \leq y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = Q(p;F)$. Since F is increasing, $F(Q(p;F)) \leq F(y_{n+1}) \leq F(y_n)$ for all $n \in \mathbb{N}$. By right-continuity of F, $\lim_{n\to\infty} F(y_n) = F(Q(p;F))$. Set $\varepsilon = \frac{1}{2}[p - F(Q(p;F))] > 0$. Hence, there eyists $N_\varepsilon \in \mathbb{N}$ such that

$$F(y_n) - F(Q(p;F)) = |F(y_n) - F(Q(p;F))| < \varepsilon = \frac{p - F(Q(p;F))}{2} \qquad \forall n \geq N_\varepsilon. \quad (4.2)$$

Therefore, $F(y_n) < \frac{1}{2}[p + F(Q(p;F))] < p$. Pick any $n \geq N_\varepsilon$, Then by construction, $y_n > Q(p;F)$ and since $F(y_n) < p$, $y_n < y$ for any y such that $F(y) \geq p$. Thus we have

$$Q(p;F) < y_n \leq \inf\{y \in \mathbb{R} : F(y) \geq p\} = Q(p;F),$$

which is a contradiction. □

Lemma 4.3. *If F is continuous at $Q(p;F)$, then $F(Q(p;F)) = p$.*

Proof of Lemma 4.3. It suffices to show continuity of F implies $F(Q(p;F)) \leq p$ since we know $F(Q(p;F)) \geq p$ for any cdf F by Lemma 4.2. We do this by contrapositive, i.e. if $F(Q(p;F)) > p$, then F has a discontinuity at $Q(p;F)$. It is sufficient to find a sequence $\{y_n\}$ such that $\lim_{n\to\infty} y_n = Q(p;F)$, but $\limsup_{n\to\infty} F(y_n) \neq F(Q(p;F))$.

Since $Q(p;F) := \inf\{y \in \mathbb{R} : F(y) \geq p\}$, we know that if $y < Q(p;F)$, then $F(y) < p$. Take a sequence $\{y_n\}$ such that $y_n \uparrow Q(p;F)$, i.e. $y_n \leq y_{n+1} < Q(p;F)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} y_n = Q(p;F)$. A concrete eyample would be for instance $y_n := Q(p;F) - (1/n)$. For each $n \in \mathbb{N}$, it follows that $F(y_n) < p$ since $y_n < Q(p;F)$. This implies $\limsup_{n\to\infty} F(y_n) \leq p < F(Q(p;F))$. It therefore follows that F is discontinuous at $Q(p;F)$. □

5 Proof of the key central limit theorem, Theorem 3.1

We will need to use Liapunov's central limit theorem (Theorem 5.1 below) in the proof of (3.2) in Theorem 3.1.

Theorem 5.1 (Liapunov's Central Limit Theorem). *For $n \in \mathbb{N}$, let $\xi_{n,1}, \ldots, \xi_{n,n}$ be independent random variables such that $\mathrm{E}[\xi_{ni}] = 0$ and for some $\delta > 0$, $\mathrm{E}\left[\xi_{ni}^{2+\delta}\right] < \infty$ for each $i \in \{1, \ldots, n\}$.*

$$\textit{If} \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \mathrm{E}\left[|\xi_{ni}|^{2+\delta}\right]}{\left(\sum_{i=1}^{n} \mathrm{Var}\left[\xi_{ni}\right]\right)^{1+(\delta/2)}} = 0, \quad \textit{then} \quad \frac{\sum_{i=1}^{n} \xi_{ni}}{\sqrt{\sum_{i=1}^{n} \mathrm{Var}\left[\xi_{ni}\right]}} \rightsquigarrow Z. \tag{5.1}$$

Proof of Theorem 5.1. This is a standard result proven in numerous texts. See for example Billingsley (1995, Theorem 27.3, p. 362) or Pollard (1984, Theorem 18 in Section 4 of Chapter III, p. 51). □

5.1 Proof of (3.2) in Theorem 3.1

The result in (3.2) arises as a consequence of the Liapunov CLT (Theorem 5.1). To that end, independently across $i = 1, \ldots, n$, let $\xi_{ni} + p_n \sim \mathrm{Bernoulli}\,(p_n)$ so that

$$\Pr\{\xi_{n,i} = 1 - p_n\} = p_n \quad \text{and} \quad \Pr\{\xi_{n,i} = -p_n\} = 1 - p_n.$$

Note that

$$\mathrm{E}\left[\xi_{n,i}\right] = 0, \quad \mathrm{E}\left[\xi_{n,i}^{2}\right] = \mathrm{Var}\left[\xi_{n,i}\right] = p_n \cdot (1 - p_n) \quad \text{and} \quad S_n - np_n \sim \sum_{i=1}^{n} \xi_{ni}.$$

Therefore, showing (3.2) is equivalent to showing

$$\text{If} \quad \lim_{n\to\infty} \frac{1}{\sqrt{np_n\,(1 - p_n)}} = 0, \quad \text{then} \quad \frac{\sum_{i=1}^{n} \xi_{ni}}{\sqrt{np_n(1 - p_n)}} \rightsquigarrow Z. \tag{5.2}$$

Since $\sum_{i=1}^{n} \mathrm{Var}\left[\xi_{ni}\right] = np_n\left(1-p_n\right)$, we can show (5.2) by proving the limit condition in the premise of (5.1) for the case $\delta = 1$. To that end

$$\sum_{i=1}^{n} \mathrm{E}\left[\left|\xi_{ni}\right|^3\right] = n\mathrm{E}\left[\left|\xi_{n1}\right|^3\right] = np_n\left(1-p_n\right)\left[p_n^2 + \left(1-p_n\right)^2\right].$$

Thus, since $p \in [0,1]$ implies $0 \le p^2 + (1-p)^2 \le 1$,

$$\frac{\sum_{i=1}^{n} \mathrm{E}\left[\left|\xi_{ni}\right|^3\right]}{\left(\sum_{i=1}^{n} \mathrm{Var}\left[\xi_{ni}\right]\right)^{3/2}} = \frac{np_n\left(1-p_n\right)\left[p_n^2 + \left(1-p_n\right)^2\right]}{\left(np_n\left(1-p_n\right)\right)^{3/2}} = \frac{p_n^2 + \left(1-p_n\right)^2}{\sqrt{np_n\left(1-p_n\right)}} \le \frac{1}{\sqrt{np_n\left(1-p_n\right)}}.$$

Hence (5.2) is proven by (5.1), and therefore (3.2) is also proven. □

5.2 Proof of (3.3) in Theorem 3.1

To prove (3.3), we will need the following ingredients.

Theorem 5.2 (Pólya's Theorem)**.** *Let $\{F_n\}$ be a sequence of cdfs and F be a cdf, all on $\mathbb{R}$, such that $F_n \rightsquigarrow F$ and F continuous. Then* $\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \left|F_n(y) - F(y)\right| = 0$.

Corollary 5.1. *Let $\{F_n\}$ be a sequence of cdfs and F be a cdf, all on $\mathbb{R}$, such that $F_n \rightsquigarrow F$ and F is continuous. If $\{x_n\}$ is a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = x \in \mathbb{R}\cup\{-\infty, +\infty\}$, then $\lim_{n\to\infty} F_n\left(x_n\right) = F(x)$.*

Proof of Theorem 5.2 and Corollary 5.1. See Section 5.2.1. □

We now prove (3.3). Let $Z_n = \left(S_n - np_n\right)/\sqrt{np_n\left(1-p_n\right)}$. Then

$$\Pr\left\{\frac{S_n}{n} \ge p\right\} = \Pr\left\{Z_n \ge \frac{\sqrt{n}\left(p-p_n\right)}{\sqrt{p_n\left(1-p_n\right)}}\right\}.$$

Since $p_n = p + n^{-1/2}\Delta + o\left(n^{-1/2}\right)$ by hypothesis,

$$\frac{\sqrt{n}\left(p-p_n\right)}{\sqrt{p_n\left(1-p_n\right)}} = -\frac{\Delta}{\sqrt{p(1-p)}} + o(1).$$

By (3.2), $Z_n \rightsquigarrow \mathrm{N}(0,1)$. Combine this with the above displays and Corollary 5.1 to Theorem 5.2 to get

$$\begin{aligned}\lim_{n\to\infty} \Pr\left\{\frac{S_n}{n} \geq p\right\} &= \lim_{n\to\infty} \Pr\left\{\frac{S_n - np_n}{\sqrt{np_n\left(1-p_n\right)}} \geq -\frac{\Delta}{\sqrt{p(1-p)}} + o(1)\right\} \\ &= 1 - \Phi\left(-\frac{\Delta}{\sqrt{p(1-p)}}\right) = \Phi\left(\frac{\Delta}{\sqrt{p(1-p)}}\right).\end{aligned}$$

□

5.2.1 Proof of Theorem 5.2 and Corollary 5.1

Corollary 5.1 is a consequence of Theorem 5.2 since

$$\begin{aligned}\left|F_n\left(x_n\right) - F(x)\right| &\leq \left|F_n\left(x_n\right) - F\left(x_n\right)\right| + \left|F\left(x_n\right) - F(x)\right| \\ &\leq \sup_{y\in\mathbb{R}} \left|F_n(y) - F(y)\right| + \left|F\left(x_n\right) - F(x)\right|.\end{aligned}$$

The first term converges to zero by Theorem 5.2. If $x \in \{+\infty, -\infty\}$, then the second term above converges to zero by standard properties of cdfs. If $x \in \mathbb{R}$, then the second term above converges to zero by continuity of F. Hence, it remains to show Theorem 5.2.

Let $k \in \mathbb{N} \setminus \{1\}$. Throughout, let $x_{k,0} = -\infty$ and $x_{k,k} = +\infty$. By continuity of F and Lemma 4.3, we can find $x_{k,1}, \ldots, x_{k,k-1}$ such that $x_{k,j-1} < x_{k,j}$ and $F\left(x_{k,j}\right) = j/k$ for each $j \in \{1, \ldots, k-1\}$. Clearly the same is true $j \in \{0, k\}$ as well. Take any $j \in \{1, \ldots, k\}$ and note that $F\left(x_{k,j}\right) = \frac{j}{k} = F\left(x_{k,j-1}\right) + \frac{1}{k}$. Thus for $x \in [x_{k,j-1}, x_{k,j}]$,

$$\begin{aligned}F_n(x) - F(x) &\leq F_n\left(x_{k,j}\right) - F\left(x_{k,j-1}\right) = F_n\left(x_{k,j}\right) - F\left(x_{k,j}\right) + \frac{1}{k} \\ &\leq \max_{l\in\{0,1,\ldots,k\}} \left|F_n\left(x_{k,l}\right) - F\left(x_{k,l}\right)\right| + \frac{1}{k}, \\ \text{and} \quad F_n(x) - F(x) &\geq F_n\left(x_{k,j-1}\right) - F\left(x_{k,j}\right) = F_n\left(x_{k,j-1}\right) - F\left(x_{k,j-1}\right) - \frac{1}{k} \\ &\geq -\max_{l\in\{0,1,\ldots,k\}} \left|F_n\left(x_{k,l}\right) - F\left(x_{k,l}\right)\right| - \frac{1}{k}.\end{aligned}$$

Therefore,

$$\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \leq \max_{l\in\{0,1,\ldots,k\}} |F_n(x_{k,l}) - F(x_{k,l})| + \frac{1}{k}. \tag{5.3}$$

Let $\varepsilon > 0$ be given. Choose $K_\varepsilon \in \mathbb{N}$ such that $K_\varepsilon \geq \max\{2, 2/\varepsilon\}$. Then $\{x_{K_\varepsilon,l}\}_{l=1}^{K_\varepsilon-1}$ are all continuity points of F since F is continuous. By $F_n \rightsquigarrow F$, $\lim_{n\to\infty} F_n(x_{K_\varepsilon,l}) = F(x_{K_\varepsilon,l})$ for each $l \in \{1,\ldots,K_\varepsilon - 1\}$. Therefore, there must exist a $N_\varepsilon \in \mathbb{N}$ such that for all $n \geq N_\varepsilon$ and all $l \in \{1,\ldots,K_\varepsilon - 1\}$, $|F_n(x_{K_\varepsilon,l}) - F(x_{K_\varepsilon,l})| \leq \varepsilon/2$. Furthermore the absolute difference is zero for $l \in \{0, K_\varepsilon\}$. Combine this with $1/K_\varepsilon \leq \varepsilon/2$ and (5.3) to see that $\sup_{x\in\mathbb{R}} |F_n(x) - F(x)| \leq \varepsilon$ for all $n \geq N_\varepsilon$. $\square$

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